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Note on a paper by Kawasaki and Sasa on Bernoulli coupled map lattices

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Abstract
In this paper we rigorously prove some statements on the symbolic dynamics for Bernoulli coupled map lattices studied by Kawasaki and Sasa. The advantage of our approach is that it is purely topological and it gives a simultaneous proof for the statements.

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1. Introduction

The objective of this paper is to give a rigorous proof for a numerically verified statement in a paper by Kawasaki and Sasa on the statistics of the Bernoulli coupled map lattices [KS]. In their paper, given a periodic symbol sequence $[s]$, they have obtained a unique periodic orbit of a coupled map lattice. However, their argument did not show that its itinerary coincides with $[s]$ (they checked this coincidence for 10^6 symbol sequences by numerical computation). The argument in the present paper gives simultaneously a proof of (i) the existence of a periodic orbit, (ii) its uniqueness and (iii) the coincidence of its itinerary with $[s]$, for a given periodic symbol sequence $[s]$. The proof uses some elementary facts on fixed point theorems for continuous maps of the interval *à la* Sharkovskii theorem.

2. Bernoulli coupled map lattices

We consider the Bernoulli coupled map lattice proposed by Sakaguchi [Sa]:

$$F : ([-1, 1] \times [-1, 1])^N \longrightarrow ([-1, 1] \times [-1, 1])^N,$$

where $F((x_0, \Delta_0), \ldots, (x_{N-1}, \Delta_{N-1})) = (f(x_0, \Delta_0), \ldots, f(x_{N-1}, \Delta_{N-1}))$ is

$$f(x_i, \Delta_i) \equiv \left(\frac{2(x_i + s_i)}{1 + s_i \Delta_i} - s_i, \tanh\left[\frac{k}{2}(s_{i-1} + s_{i+1}) \right] \right),$$

and k is a positive parameter.

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The multi-valued itinerary map

$$\pi : ([-1, 1] \times [-1, 1])^N \longrightarrow \{+1, -1\}^N$$

is given by $\pi((x_0, \Delta_0), \ldots, (x_{N-1}, \Delta_{N-1})) \equiv (s_0, \ldots, s_{N-1})$, where

$$s_i \equiv \begin{cases} +1 & (-1 \leqslant x_i \leqslant \Delta_i \leqslant 1), \\ -1 & (-1 \leqslant \Delta_i \leqslant x_i \leqslant 1). \end{cases}$$

Here, note that the definition of the region $\{-1 \leqslant x_i \leqslant \Delta_i \leqslant 1\}$ corresponding to $s_i = +1$ is modified from that in [KS] so that it becomes a compact set. This is the reason why our itinerary map becomes multi-valued. However, the definition of the itinerary coincides with the previous one whenever a periodic orbit does not touch the diagonal $\{x = \Delta\}$.

We write $X^{n+1} = F(X^n)$ for $X^n = \left(\left(x_0^n, \Delta_0^n\right), \ldots, \left(x_{N-1}^n, \Delta_{N-1}^n\right)\right)$ and $[s] = (s^0, \ldots, s^{p-1}) \in (\{+1, -1\}^N)^p$. Our main claim in this paper is

Theorem 2.1. *Given a set of p symbol sequences $[s] \in (\{+1, -1\}^N)^p$, there exists a unique $X_0 \in ([-1, 1] \times [-1, 1])^N$ so that $f^p(X^0) = X^0$ and $[s] = (\pi(X^0), \ldots, \pi(X^{p-1}))$.*

This theorem gives not only the existence and the uniqueness of a periodic orbit (which has been already shown in [KS]) but also the coincidence of its itinerary with a given periodic symbol sequence $[s]$.

3. Fixed point theorems on the interval

We here summarize some preliminary results on fixed point theorems for continuous maps of the interval. Let $I \subset \mathbb{R}$ be a closed interval and $f : I \to \mathbb{R}$ be a continuous map. Given two subintervals $J_1 \subset I$ and $J_2 \subset \mathbb{R}$, we say that J_1 *covers* J_2 *by* f if $f(J_1) \supset J_2$ holds. Our first basic observation is as follows.

Lemma 3.1. *Assume that $f : I \to \mathbb{R}$ is continuous. If I covers itself by f, i.e. $f(I) \supset I$, then there exists $x_0 \in I$ so that $f(x_0) = x_0$. Moreover, if f is C^1 and $|f'(x)| > 1$ for all $x \in I$, then such fixed point x_0 is unique.*

Proof. Let us write $I = [\alpha, \beta]$. Since I covers itself by f, both $A = f^{-1}(\alpha)$ and $B = f^{-1}(\beta)$ are non-empty subset of I. Take $a \in A$ and $b \in B$. Without loss of generality, we may assume that $a < b$. Define $g(x) = f(x) - x$. Then $g(a) = f(a) - a = \alpha - a \leqslant 0$ and $g(b) = f(b) - b = \beta - b \geqslant 0$. By the intermediate value theorem, there is $x_0 \in [a, b]$ so that $g(x_0) = x_0$, i.e. $f(x_0) = x_0$.

Now assume that f is C^1 and $|f'(x)| > 1$ for all $x \in I$. If there were two distinct fixed points of f in I, the mean value theorem would assert the existence of $x_1 \in I$ with $f'(x_1) = 1$, a contradiction. □

Lemma 3.2. *Assume that $f : I \to \mathbb{R}$ is continuous. Let J be a closed subinterval of $\mathbb{R}$. If I covers J by f, then there exists a subinterval $K \subset I$ so that $f(K) = J$.*

Proof. Let us write $J = [\alpha, \beta]$. Since I covers J by f, both $A = f^{-1}(\alpha)$ and $B = f^{-1}(\beta)$ are non-empty subset of I. Evidently, A and B are disjoint and compact in I, thus $d_0 \equiv d(A, B) > 0$. Moreover, there are $a \in A$ and $b \in B$ so that $d_0 = d(a, b)$. Without loss of generality, we may assume that $a < b$. Put $K \equiv [a, b]$. Then, $f(K) \supset J$ follows from the intermediate value theorem since $f(a) = \alpha$ and $f(b) = \beta$. If there were $c \in (a, b)$ so that $f(c) \geqslant \beta$, then there would be an element of B in $(a, c]$ by the intermediate value theorem. However, the open interval (a, b) should be disjoint from both A and B by the definition of

$d_0 = d(A, B)$, a contradiction. Thus, for all $c \in (a, b)$ we have $f(c) < \beta$. A similar argument shows that $f(c) > \alpha$ for all $c \in (a, b)$. This implies $f(K) \subset J$. □

By using this fact, the first lemma can be extended to a non-autonomous periodic case in the following way.

Proposition 3.3. *Assume that there exists a sequence of closed subintervals $J_0, J_1, \ldots, J_p \equiv J_0$ of I and a sequence of continuous maps $f_i : J_i \to \mathbb{R}$ such that J_i covers J_{i+1} by f_i for $i = 0, 1, \ldots, p-1$. Then, there exists $x_0 \in J_0$ so that $f_{p-1} \circ \cdots \circ f_0(x_0) = x_0$ and $f_{i-1} \circ \cdots \circ f_0(x_0) \in J_i$ for $i = 1, 2, \ldots, p$. Moreover, if f_i is C^1 and $|f_i'(x)| > 1$ for all $x \in J_i$ and all i, then such point x_0 is unique.*

Proof. First consider $f_{p-1} : J_{p-1} \to \mathbb{R}$. Since J_{p-1} covers $J_p = J_0$ by f_{p-1}, there is a closed subinterval $K_{p-1} \subset J_{p-1}$ so that $f_{p-1}(K_{p-1}) = J_0$ by lemma 3.2. Since J_{p-2} covers J_{p-1} by f_{p-2} and since $K_{p-1} \subset J_{p-1}$, there is a closed subinterval $K_{p-2} \subset J_{p-2}$ so that $f_{p-2}(K_{p-2}) = K_{p-1}$, thus $f_{p-1} \circ f_{p-2}(K_{p-2}) = J_0$. We can inductively find a sequence of closed subintervals $K_j \subset J_i$ so that $f_i(K_i) = K_{i+1}$, thus $f_{p-1} \circ \cdots \circ f_{p-n}(K_{p-n}) = J_0$. In particular, one gets $f_{p-1} \circ \cdots \circ f_0(K_0) = J_0 \supset K_0$, and this means that K_0 covers itself by $f_{p-1} \circ \cdots \circ f_0$. By lemma 3.1, we know the existence of $x_0 \in K_0$ with $f_{p-1} \circ \cdots \circ f_0(x_0) = x_0$. By the construction of K_i, it also follows that $f_{i-1} \circ \cdots \circ f_0(x_0) \in K_i \subset J_i$ for $i = 1, 2, \ldots, p$. If f_i is C^1 and $|f_i'(x)| > 1$ for all $x \in J_i$, then $\bigcap_{i=0}^{p-1} f_0^{-1} \circ \cdots \circ f_{i-1}^{-1}(J_i)$ becomes a closed interval. Now, the uniqueness part can be proved as in lemma 3.1, and hence we are done. □

4. Proof of theorem 2.1

Assume that a number p and symbol sequences $[s]$ are given and fixed. As was pointed out by [KS], the sequences $[s]$ uniquely determine as $\Delta_i^n = \tanh\left[\frac{k}{2}(s_{i-1} + s_{i+1})\right]$ for $n = 0, 1, \ldots, p-1$ and $i = 0, 1, \ldots, N-1$. This in particular means that our task is to study the iteration of a family of one-dimensional maps:

$$f_n(\cdot) \equiv f\left(\cdot, \Delta_i^n\right) : [-1, 1] \times \left\{\Delta_i^n\right\} \longrightarrow \mathbb{R} \times \left\{\Delta_i^{n+1}\right\}$$

($n = 0, 1, \ldots, p-1$) for each i, where Δ_i^n and Δ_i^{n+1} are fixed. Note that, since p and N are finite, there exists $\varepsilon > 0$ independent of n and i so that the distance between Δ_i^n and the boundary $\{-1, 1\} = \partial[-1, 1]$ is at least ε.

Put $T_{+1} = \{(x, \Delta) : -1 \leqslant x \leqslant \Delta \leqslant 1\}$ and $T_{-1} = \{(x, \Delta) : -1 \leqslant \Delta \leqslant x \leqslant 1\}$, and let $I_\sigma(\delta) = T_\sigma \cap \{\Delta = \delta\}$ ($\sigma = +1, -1$) be a closed interval. We first show that $I_{s_i}(\delta)$ covers $I_{s_i'}(\delta')$ by $f(\cdot, \delta) : I_{s_i}(\delta) \to I_{s_i'}(\delta')$ for any choice of s_i and s_i', where $\delta' = \tanh[\frac{k}{2}(s_{i-1} + s_{i+1})]$. To see this, assume first that $s_i = +1$. Then, the x-coordinate of $f(-1, \delta)$ is computed as

$$\frac{2(x + s_i)}{1 + s_i\delta} - s_i = \frac{2(-1 + 1)}{1 + \delta} - 1 = -1$$

for $-1 \leqslant \delta \leqslant 1$. Similarly, the x-coordinate of $f(\delta, \delta)$ is

$$\frac{2(x + s_i)}{1 + s_i\delta} - s_i = \frac{2(\delta + 1)}{1 + \delta} - 1 = 1$$

for $-1 \leqslant \delta \leqslant 1$. This shows the desired claim for the case $s_i = +1$. The case $s_i = -1$ is similar. Now we apply proposition 3.3 to get an initial point x_i^0 so that $f_{p-1} \circ \cdots \circ f_0\left(x_i^0\right) = x_i^0$ and $f_{j-1} \circ \cdots \circ f_0\left(x_i^0\right) \in I_{s_j}(\delta_j')$ for $j = 1, 2, \ldots, p$. By putting $X^0 \equiv \left(\left(x_0^0, \Delta_0^0\right), \ldots, \left(x_{N-1}^0, \Delta_{N-1}^0\right)\right)$, this means that $f^p(X^0) = X^0$ and $[s] \in (\pi(X^0), \ldots, \pi(X^{p-1}))$.

If we return to the original definition $\{-1 \leqslant x_i < \Delta_i \leqslant 1\}$ of the region corresponding to $s_i = +1$, then the orbit of a point in the diagonal by f becomes $(\delta^0, \delta^0) \mapsto (-1, \delta^1) \mapsto (-1, \delta^2) \mapsto \cdots$, which cannot be periodic (note that $|\delta^1| < 1$, so the point $(-1, -1)$ is not periodic). Thus, any periodic point does not touch the diagonal and we conclude that $[s] = (\pi(X^0), \ldots, \pi(X^{p-1}))$.

As already remarked, once p and $[s]$ are fixed, there exists $\varepsilon > 0$ so that $\Delta_i^n \in [-1 + \varepsilon, 1 - \varepsilon]$ for all n and i. Hence,

$$\left|\frac{\partial}{\partial x} f_1(x, \Delta_i^n)\right| = \left|\frac{2}{1 + s_i \Delta_i^n}\right| \geqslant \frac{2}{2 - \varepsilon} > 1,$$

where $f_1(x, \Delta_i^n) = \frac{2(x_i + s_i)}{1 + s_i \Delta_i} - s_i$ is the x-coordinate of $f\big(x, \Delta_i^n\big)$. We can now apply the uniqueness part of proposition 3.3 to finish the proof of theorem 2.1.

Remark 4.1. Since the proof of proposition 3.3 is purely topological and does not depend on the specific form of the Bernoulli coupled map lattice, one can apply proposition 3.3 for more general class of systems to obtain a similar result as theorem 2.1.

References

[KS] Kawasaki M and Sasa S 2005 *Phys. Rev.* E **72** 037202, 4
[Sa] Sakaguchi H 1988 *Prog. Theor. Phys.* **80** 7